

# A COMMON FIXED POINT THEOREM FOR A COMMUTING FAMILY OF WEAK\* CONTINUOUS NONEXPANSIVE MAPPINGS

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**ABSTRACT.** It is shown that if  $\mathcal{S}$  is a commuting family of weak\* continuous nonexpansive mappings acting on a weak\* compact convex subset  $C$  of the dual Banach space  $E$ , then the set of common fixed points of  $\mathcal{S}$  is a nonempty nonexpansive retract of  $C$ . This partially solves a long-standing open problem in metric fixed point theory in the case of commutative semigroups.

## 1. INTRODUCTION

A subset  $C$  of a Banach space  $E$  is said to have the fixed point property if every nonexpansive mapping  $T : C \rightarrow C$  (i.e.,  $\|Tx - Ty\| \leq \|x - y\|$  for  $x, y \in C$ ) has a fixed point. A general problem, initiated by the works of F. Browder, D. Göhde and W. A. Kirk and studied by numerous authors for over 40 years, is to classify those  $E$  and  $C$  which have the fixed point property. For a fuller discussion on this topic we refer the reader to [3, 6].

In this paper we concentrate on weak\* compact convex subsets of the dual Banach space  $E$ . In 1976, L. Karlovitz (see [5]) proved that if  $C$  is a weak\* compact convex subset of  $\ell_1$  (as the dual to  $c_0$ ) then every nonexpansive mapping  $T : C \rightarrow C$  has a fixed point. His result was extended by T.C. Lim [11] to the case of left reversible topological semigroups. On the other hand, C. Lennard showed the example of a weak\* compact convex subset of  $\ell_1$  with the weak\* topology induced by its predual  $c$  and an affine contractive mapping without fixed points (see [12, Example 3.2]). This shows that, apart from nonexpansiveness, some additional assumptions have to be made to obtain the fixed points.

Let  $S$  be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each  $t \in S$ , the mappings  $s \rightarrow t \cdot s$  and  $s \rightarrow s \cdot t$  from  $S$  into  $S$  are continuous. Consider the following fixed point property:

*(F<sub>\*</sub>): Whenever  $\mathcal{S} = \{T_s : s \in S\}$  is a representation of  $S$  as norm-nonexpansive mappings on a non-empty weak\* compact convex set  $C$  of a dual Banach space  $E$  and the mapping  $(s, x) \rightarrow T_s(x)$  from  $S \times C$  to  $C$  is jointly continuous, where  $C$  is equipped with the weak\* topology of  $E$ , then there is a common fixed point for  $\mathcal{S}$  in  $C$ .*

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It is not difficult to show (see, e.g., [9, p. 528]) that property  $(F_*)$  implies that  $S$  is left amenable (in the sense that  $LUC(S)$ , the space of bounded complex-valued left uniformly continuous functions on  $S$ , has a left invariant mean). Whether the converse is true is a long-standing open problem, posed by A. T.-M. Lau in [8] (see also [9, Problem 2], [10, Question 1]).

It is well known that all commutative semigroups are left amenable. The aim of this paper is to give a partial answer to the above problem by showing that every commuting family  $\mathcal{S}$  of weak\* continuous nonexpansive mappings acting on a weak\* compact convex subset  $C$  of the dual Banach space  $E$  has common fixed points. Moreover, we prove that the set  $\text{Fix } \mathcal{S}$  of fixed points is a nonexpansive retract of  $C$ .

Note that the structure of  $\text{Fix } \mathcal{S}$  (with  $\mathcal{S}$  commutative) was examined by R. Bruck (cf. [1, 2]) who proved that if every nonexpansive mapping  $T : C \rightarrow C$  has a fixed point in every nonempty closed convex subset of  $C$  which is invariant under  $T$ , and  $C$  is convex and weakly compact or separable, then  $\text{Fix } \mathcal{S}$  is a nonexpansive retract of  $C$ . We are able to mix the elements of Bruck's method with some properties of  $w^*$ -continuous and nonexpansive mappings to get the desired result.

## 2. PRELIMINARIES

Let  $E$  be the dual of a Banach space  $E_*$ . In this paper we focus on the weak\* topology – the smallest one satisfying the condition: for all  $e \in E$ , the functional  $\hat{e}(x) = x(e)$  is continuous (in the strong topology). This definition opens up the possibility to consider the so-called weak\* properties, for example,  $w^*$ -compactness (compactness in the  $w^*$ -topology),  $w^*$ -completeness, etc. In this topology,  $E$  becomes a locally convex Hausdorff space. We say that a dual Banach space  $E$  has the  $w^*$ -FPP if every nonexpansive self-mapping defined on a nonempty  $w^*$ -compact convex subset of  $E$  has a fixed point. It is known that  $\ell_1 = c_0^*$  and some other Banach lattices have  $w^*$ -FPP, however  $\ell_1 = c^*$  as well as the duals of  $C(\Omega)$ , where  $\Omega$  is an infinite compact Hausdorff topological space, do not possess this property.

A non-void set  $D \subset C$  is said to be a nonexpansive retract of  $C$  if there exists a nonexpansive retraction  $R : C \rightarrow D$  (i.e., a nonexpansive mapping  $R : C \rightarrow D$  such that  $R|_D = I$ ). Since we deal a lot with  $w^*$ -continuous nonexpansive mappings, we abbreviate them to  $w^*$ -CN.

We conclude with recalling the following consequence of the Ishikawa theorem (see [4]): if  $C$  is a bounded convex subset of a Banach space  $X$ ,  $\gamma \in (0, 1)$ , and  $T : C \rightarrow C$  is nonexpansive, then the mapping  $T_\gamma = (1 - \gamma)I + \gamma T$  is asymptotically regular, i.e.,  $\lim_{n \rightarrow \infty} \|T_\gamma^{n+1}x - T_\gamma^n x\| = 0$  for every  $x \in C$ . We use this theorem in Lemma 3.4.

## 3. FIXED-POINT THEOREMS

We begin with a structural result concerning a single  $w^*$ -continuous nonexpansive mapping  $T : C \rightarrow C$ .

**Theorem 3.1.** *Let  $C$  be a nonempty weak\* compact convex subset of the dual Banach space. Then for any  $w^*$ -CN self-mapping  $T$  on  $C$ , the set  $\text{Fix } T$  of fixed points of  $T$  is a (nonempty) nonexpansive retract of  $C$ .*

The proof will follow by constructing gradually (and establishing properties of) three functions, each one defined in the means of the earlier, and the last one being the retraction from  $C$  to  $\text{Fix } T$ .

*Proof.* Notice first that  $C$  is complete in the strong topology. Now, for  $x \in C$  and a postive integer  $n$ , consider a mapping  $T_x : C \rightarrow C$  defined by

$$T_x z = \frac{1}{n}x + \left(1 - \frac{1}{n}\right) Tz, \quad z \in C.$$

It is not difficult to see that  $T_x$  is a contraction:

$$\|T_x y - T_x z\| \leq \left(1 - \frac{1}{n}\right) \|y - z\|.$$

Hence and from completeness of  $C$ , it follows from the Banach Contraction Principle that there exists exactly one point  $F_n x \in C$  such that  $T_x F_n x = F_n x$ . This defines a mapping  $F_n : C \rightarrow C$  by

$$F_n x = \frac{1}{n}x + \left(1 - \frac{1}{n}\right) T F_n x \quad (1)$$

for  $x \in C$ . Thus

$$\|T F_n x - F_n x\| = \frac{1}{n} \|T F_n x - x\| \leq \frac{1}{n} \text{diam } C$$

and consequently,

$$\lim_n \|T F_n x - F_n x\| = 0$$

since  $C$  is bounded in norm as a weak\* compact subset of a Banach space.

Notice that for  $x \in \text{Fix } T$  we have

$$T_x x = x$$

and consequently  $F_n x = x$ .

Furthermore,  $F_n x$  is nonexpansive. Indeed,

$$F_n x - F_n y = T_x F_n x - T_y F_n y = \frac{1}{n}(x - y) + \left(1 - \frac{1}{n}\right) (Tx - Ty) \quad (2)$$

which, by putting it into norm and using the triangle inequality and non-expansiveness of  $T$ , gives us a desired statement.

Notice that we can view  $C^C$  as the product space of copies of  $C$ , where each copy is endowed with the  $w^*$ -topology. Then, according to Tychonoff's theorem,  $C^C$  is compact in the product topology generated in this way ("w\*-product topology"). It follows that a sequence  $(F_n)_{n \in \mathbb{N}}$  of elements from  $C^C$  has a convergent subnet  $(F_{n_\alpha})_{\alpha \in \Lambda}$  and we can define

$$R = w^* - \lim_{\alpha} F_{n_\alpha},$$

where the above limit should be understood as taken in the aforementioned  $w^*$ -product topology. Now we can treat the application of  $R$  to some  $x \in C$  as the projection of the mapping onto the  $x$ -th coordinate and since such projections are continuous in the product topology, we obtain

$$Rx = w^*\text{-}\lim_{\alpha} F_{n_{\alpha}}x,$$

where this limit is an ordinary  $w^*$ -limit. With this approach, we are able to construct one subnet which guarantees convergence for all  $x \in C$ .

Notice that

$$TRx = w^*\text{-}\lim_{\alpha} TF_{n_{\alpha}}x$$

since  $T$  is weak\* continuous. Now, it follows from the weak\* lower semicontinuity of the norm that for any  $x \in C$ ,

$$\|TRx - Rx\| = \left\| w^*\text{-}\lim_{\alpha} (TF_{n_{\alpha}}x - F_{n_{\alpha}}x) \right\| \leq \liminf_{\alpha} \|TF_{n_{\alpha}}x - F_{n_{\alpha}}x\| = 0$$

and hence

$$TRx = Rx$$

which means that  $Rx \in \text{Fix } T$ . Furthermore,  $Rx = x$  if  $x \in \text{Fix } T$ .

We can now use (2) and the weak\* lower semicontinuity of the norm to prove that  $R$  is nonexpansive:

$$\begin{aligned} \|Rx - Ry\| &= \left\| w^*\text{-}\lim_{\alpha} (F_{n_{\alpha}}x - F_{n_{\alpha}}y) \right\| \\ &\leq \liminf_{\alpha} \left\| \frac{1}{n_{\alpha}}(x - y) + \left(1 - \frac{1}{n_{\alpha}}\right)(Tx - Ty) \right\| \leq \limsup_{\alpha} \frac{1}{n_{\alpha}} \|x - y\| \\ &\quad + \limsup_{\alpha} \left(1 - \frac{1}{n_{\alpha}}\right) \|Tx - Ty\| = \|Tx - Ty\| \leq \|x - y\|. \end{aligned}$$

Thus we conclude that  $\text{Fix } T$  is indeed a nonexpansive retract of  $C$ .  $\square$

*Remark 3.2.* The  $w^*$ -continuity of  $T$  cannot be omitted in the assumptions of Theorem 3.1. Indeed, otherwise we would conclude that any dual Banach space has  $w^*$ -FPP. But it is known (see, e.g., [12, Example 3.2]) that  $\ell_1$  (as the dual to the Banach space  $c$ ) fails the  $w^*$ -FPP, a contradiction.

The following example shows that we would not be able to relax the assumption of the nonexpansiveness of  $T$  to continuity, either, even if we only postulated the existence of a (continuous) retraction.

**Example 1.** Let  $\ell_1 = c_0^*$  and define

$$T(x_1, x_2, x_3, \dots) = ((x_1)^2, 0, x_2, x_3, \dots)$$

on the unit ball  $B_{\ell_1}$ . Notice that  $T : B_{\ell_1} \rightarrow B_{\ell_1}$  is  $w^*$ -continuous and  $\text{Fix } T = \{(\pm 1, 0, \dots)\}$ . But a non-connected set cannot be a retract of the ball.

Our next objective is to generalize Theorem 3.1 to a commuting family of  $w^*$ -continuous nonexpansive mappings. If  $\mathcal{S} = \{T_s : s \in S\}$  is a family of mappings, we denote by

$$\text{Fix } \mathcal{S} = \bigcap_{s \in S} \text{Fix } T_s$$

the set of common fixed points of  $\mathcal{S}$ .

We first prove a lemma which resembles [1, Lemma 6].

**Lemma 3.3.** *Let  $\mathcal{S}$  be a family of commuting self-mappings acting on a set  $C$  and suppose that there exists a retraction  $R$  of  $C$  onto  $\text{Fix } \mathcal{S}$ . If  $\tilde{T}$  commutes with every element of the family  $\mathcal{S}$ , then*

$$\text{Fix } \mathcal{S} \cap \text{Fix } \tilde{T} = \text{Fix}(\tilde{T}R).$$

*Proof.* The inclusion from left to right follows from the simple observation that if  $x \in \text{Fix } \mathcal{S} \cap \text{Fix } \tilde{T}$ , then  $Rx = x$  and  $\tilde{T}x = x$ .

For the other direction, assume  $x \in \text{Fix}(\tilde{T}R)$  which means  $\tilde{T}Rx = x$ . Then, for every  $T \in \mathcal{S}$ , it follows from the commutativity and the fact that  $Rx \in \text{Fix } T$  that

$$T\tilde{T}Rx = \tilde{T}(TRx) = \tilde{T}Rx.$$

Therefore  $\tilde{T}Rx \in \text{Fix } T$  for every  $T \in \mathcal{S}$  and consequently

$$x = \tilde{T}Rx \in \text{Fix } \mathcal{S}.$$

Since  $R$  is a retraction onto  $\text{Fix } \mathcal{S}$ , we have  $Rx = x$  and hence  $\tilde{T}x = x$ . It follows that  $x \in \text{Fix } \mathcal{S} \cap \text{Fix } \tilde{T}$  which proves the inclusion and the whole lemma.  $\square$

**Lemma 3.4.** *Suppose that  $C$  is as in Theorem 3.1 and  $\mathcal{S}_n = \{T_1, \dots, T_n\}$  is a finite commuting family of  $w^*$ -CN self-mappings on  $C$ . Then  $\text{Fix } \mathcal{S}_n$  is a nonexpansive retract of  $C$ .*

*Proof.* We will show by induction on  $n$  that there exists a nonexpansive retraction  $R_n$  from  $C$  onto  $\text{Fix } \mathcal{S}_n$ . The base case  $n = 1$  follows directly from Theorem 3.1 since  $\text{Fix } \mathcal{S}_1 = \text{Fix } T_1$ .

Now assume that there exists a nonexpansive retraction  $R_n$  of  $C$  onto  $\text{Fix } \mathcal{S}_n$ . We need to show the existence of a nonexpansive retraction  $R_{n+1}$  of  $C$  onto  $\text{Fix } \mathcal{S}_{n+1}$ .

Let

$$\tilde{R}_n x = \frac{1}{2}x + \frac{1}{2}T_{n+1}R_n x, \quad x \in C,$$

and consider a sequence  $(\tilde{R}_n^k)_{k \in \mathbb{N}}$  of successive iterations of  $\tilde{R}_n$ . As in the proof of Theorem 3.1, we can view  $C^C$  as the product space, compact with respect to the  $w^*$ -topology on  $C$ . Hence the sequence  $(\tilde{R}_n^k)_{k \in \mathbb{N}}$  has a convergent subnet  $(\tilde{R}_n^{k_\alpha})_{\alpha \in \Lambda}$  and we can define

$$R_{n+1}x = w^*-\lim_{\alpha} \tilde{R}_n^{k_\alpha} x$$

for every  $x \in C$ .

Since  $T_{n+1}R_n$  is nonexpansive as a composition of such mappings, it is easy to see that also  $\tilde{R}_n$  is nonexpansive. The nonexpansiveness of  $R_{n+1}$  now follows from the weak\* lower semicontinuity of the norm. It is also easy to see that  $\text{Fix } T_{n+1}R_n \subset \text{Fix } R_{n+1}$  and, by using Lemma 3.3, we conclude that

$$\text{Fix } \mathcal{S}_{n+1} \subset \text{Fix } R_{n+1}.$$

But this still does not prove that  $R_{n+1}$  is a mapping we are looking for, nor that  $\text{Fix } \mathcal{S}_{n+1}$  is nonempty. To complete the proof, we must show that  $R_{n+1}$  is a mapping onto  $\text{Fix } \mathcal{S}_{n+1}$ . The rest of the proof is about showing this fact.

Since  $C$  is convex closed and bounded, and  $\tilde{R}_n$  is the convex combination of a nonexpansive mapping and the identity, it follows from the Ishikawa theorem [4] that  $\tilde{R}_n$  is asymptotically regular, i.e.,

$$\lim_{k \rightarrow \infty} \left\| \tilde{R}_n^{k+1}x - \tilde{R}_n^kx \right\| = 0$$

for every  $x \in C$ .

Now, fix  $x$  and notice that  $(\tilde{R}_n^{k_\alpha}x)_{\alpha \in \Lambda}$  is an approximate fixed point net for the mapping  $T_{n+1}R_n$ . To see this, use the equation

$$\tilde{R}_n^{k_\alpha+1}x = \frac{1}{2} \left( \tilde{R}_n^{k_\alpha}x - T_{n+1}R_n \tilde{R}_n^{k_\alpha}x \right) + T_{n+1}R_n \tilde{R}_n^{k_\alpha}x$$

and the asymptotical regularity in the following calculations:

$$\begin{aligned} \limsup_{\alpha} \left\| T_{n+1}R_n \tilde{R}_n^{k_\alpha}x - \tilde{R}_n^{k_\alpha}x \right\| &\leq \limsup_{\alpha} \left\| T_{n+1}R_n \tilde{R}_n^{k_\alpha}x - \tilde{R}_n^{k_\alpha+1}x \right\| \\ &+ \lim_{\alpha} \left\| \tilde{R}_n^{k_\alpha+1}x - \tilde{R}_n^{k_\alpha}x \right\| = \limsup_{\alpha} \left\| T_{n+1}R_n \tilde{R}_n^{k_\alpha}x - \tilde{R}_n^{k_\alpha+1}x \right\| \\ &= \frac{1}{2} \limsup_{\alpha} \left\| T_{n+1}R_n \tilde{R}_n^{k_\alpha}x - \tilde{R}_n^{k_\alpha}x \right\|. \end{aligned}$$

Thus we conclude that

$$\lim_{\alpha} \left\| T_{n+1}R_n \tilde{R}_n^{k_\alpha}x - \tilde{R}_n^{k_\alpha}x \right\| = 0, \quad (3)$$

as desired.

Now, for brevity, denote  $r_\alpha = \tilde{R}_n^{k_\alpha}x$  and notice that for every  $m \leq n$

$$T_m T_{n+1} R_n r_\alpha = T_{n+1} T_m R_n r_\alpha = T_{n+1} R_n r_\alpha.$$

That is,  $T_{n+1}R_n r_\alpha \in \text{Fix } T_m$  which is equivalent to the statement that  $T_{n+1}R_n r_\alpha$  belongs to  $\text{Fix } \mathcal{S}_n$ . It follows that

$$T_{n+1}R_n r_\alpha = R_n T_{n+1} R_n r_\alpha.$$

and using the equation (3), we obtain

$$\begin{aligned} \limsup_{\alpha} \|R_n r_\alpha - r_\alpha\| &\leq \limsup_{\alpha} \|R_n r_\alpha - T_{n+1}R_n r_\alpha\| + \lim_{\alpha} \|T_{n+1}R_n r_\alpha - r_\alpha\| \\ &= \limsup_{\alpha} \|R_n r_\alpha - R_n T_{n+1} R_n r_\alpha\| \leq \lim_{\alpha} \|r_\alpha - T_{n+1} R_n r_\alpha\| = 0. \end{aligned} \quad (4)$$

In the same manner we can see that for every  $m \leq n$ ,

$$\begin{aligned} \limsup_{\alpha} \|T_m r_{\alpha} - r_{\alpha}\| &\leq \limsup_{\alpha} \|T_m r_{\alpha} - T_m R_n r_{\alpha}\| + \limsup_{\alpha} \|T_m R_n r_{\alpha} - r_{\alpha}\| \\ &\leq \lim_{\alpha} \|r_{\alpha} - R_n r_{\alpha}\| + \lim_{\alpha} \|R_n r_{\alpha} - r_{\alpha}\| = 0. \end{aligned}$$

Since  $T_m$  is  $w^*$ -continuous, this easily yields

$$T_m R_{n+1} x = R_{n+1} x$$

and, consequently,

$$R_{n+1} x \in \text{Fix } \mathcal{S}_n. \quad (5)$$

Finally, by using (3) and (4), we get

$$\begin{aligned} \limsup_{\alpha} \|T_{n+1} r_{\alpha} - r_{\alpha}\| &\leq \limsup_{\alpha} \|T_{n+1} r_{\alpha} - T_{n+1} R_n r_{\alpha}\| + \lim_{\alpha} \|T_{n+1} R_n r_{\alpha} - r_{\alpha}\| \\ &+ \lim_{\alpha} \|T_{n+1} R_n r_{\alpha} - r_{\alpha}\| \leq \lim_{\alpha} \|r_{\alpha} - R_n r_{\alpha}\| = 0. \end{aligned}$$

Then, from the  $w^*$ -continuity of  $T_{n+1}$ ,

$$T_{n+1} R_{n+1} x = R_{n+1} x$$

which combined with (5), gives

$$R_{n+1} x \in \text{Fix } \mathcal{S}_{n+1}.$$

That is,  $\text{Fix } \mathcal{S}_{n+1}$  is nonempty and  $R_{n+1}$  acts onto it, which completes the proof.  $\square$

We are now in a position to prove our main theorem.

**Theorem 3.5.** *Suppose that  $C$  is as in Theorem 3.1 and  $\mathcal{S}$  is an arbitrary family of commuting  $w^*$ -CN self-mappings on  $C$ . Then  $\text{Fix } \mathcal{S}$  is a nonexpansive retract of  $C$ .*

*Proof.* If  $\mathcal{S}$  is finite, we can use lemma 3.4. So assume that  $\mathcal{S}$  is infinite. First notice that

$$\text{Fix } T = (T - I)^{-1} \{0\}$$

is closed in the  $w^*$ -topology for every  $T \in \mathcal{S}$ . Let

$$\Lambda = \{\alpha \subset \mathcal{S} : \#\alpha < \infty\}$$

be a directed set with the inclusion relation  $\leq$ . Denote by  $R_{\alpha}$  the nonexpansive retraction from  $C$  to  $\text{Fix}_{\alpha} = \bigcap_{T \in \alpha} \text{Fix } T$  (a more convenient way of writing  $\text{Fix } \alpha$ ) which existence is guaranteed by Lemma 3.4. Then we have a net  $(R_{\alpha})_{\alpha \in \Lambda}$ , and we can select a subnet  $(R_{\alpha_{\gamma}})_{\gamma \in \Gamma}$ ,  $w^*$ -convergent for any  $x \in C$ . Define

$$Rx = w^* - \lim_{\gamma} R_{\alpha_{\gamma}} x.$$

For a fixed  $T \in \mathcal{S}$ , take  $\gamma_0$  such that  $\alpha_{\gamma} \geq \{T\}$  for every  $\gamma \geq \gamma_0$ . It exists, straightforward from the subnet definition. Then

$$\forall \gamma \geq \gamma_0 \quad R_{\alpha_{\gamma}} x \in \text{Fix}_{\alpha_{\gamma}} \subset \text{Fix}_{\alpha_{\gamma_0}} \subset \text{Fix } T$$

and hence  $R_{\alpha_\gamma}x$  lies eventually in the  $w^*$ -closed set  $\text{Fix } T$ . Therefore,  $Rx \in \text{Fix } T$  for every  $T \in \mathcal{S}$  which implies  $Rx \in \text{Fix } \mathcal{S}$ . It is easy to see that  $R$  is nonexpansive. Also, for every  $\alpha$ ,

$$x \in \text{Fix } \mathcal{S} \implies x \in \text{Fix } \alpha \implies R_\alpha x = x,$$

from which follows

$$Rx = x, x \in \text{Fix } \mathcal{S}. \quad (6)$$

Thus  $R$  is a nonexpansive retraction from  $C$  onto  $\text{Fix } \mathcal{S}$ .  $\square$

*Remark 3.6.* In particular, the set  $\text{Fix } \mathcal{S}$  is non-empty. Thus Theorem 3.5 answers affirmatively [10, Question 1] in the case of commutative semi-groups.

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